Self-similar Bianchi type VIII and IX models

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Abstract

It is shown that in transitively self-similar spatially homogeneous tilted perfect fluid models the symmetry vector is not normal to the surfaces of spatial homogeneity. A direct consequence of this result is that there are no self-similar Bianchi VIII and IX tilted perfect fluid models. Furthermore the most general Bianchi VIII and IX spacetime which admit a four dimensional group of homotheties is given.

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1   Introduction

Despite the fact that in Spatially Homogeneous (SH) models the field equations are reduced to a system of ordinary differential equations, not many exact solutions are known, especially in the case of tilted perfect fluid models. This has initiated the study of these models using the methods of the theory of dynamical systems, where one examines their behaviour from a qualitative point of view and, in particular, at early, late and intermediate periods of their history. These studies have revealed that *transitively self-similar* SH models act as early (i.e. near to the initial singularity) and late time asymptotic states for more general spatially homogeneous models [1]. However it has been pointed out that there is an open set of SH models which are not asymptotically self-similar i.e. *may not* admit proper HVF (although, in some cases, their asymptotic states can be successively approximated by an infinite sequence of self-similar models). For these and other reasons it is of primary interest to determine all SH perfect fluid models which are or not *transitively self-similar.*

We recall that a self-similar model admits a Homothetic Vector Field (HVF) **H** and is defined by the requirement:

$$\mathcal{L}_{\mathbf{H}}g_{ab} = 2\psi g_{ab} \tag{1}$$

where ψ=const. is the homothetic factor.

Self-similar SH vacuum and non-tilted perfect fluid models have been determined by Hsu and Wainwright [2] who proved that Bianchi type VIII and IX models are not self-similar [1, 2]. Concerning the tilted perfect fluid models, Bradley [3] has stated that there

do not exist tilted dust self-similar models whereas Hewitt [4] has found the general self-similar Bianchi type II solution and Rosquist and Jantzen have determined a rotational tilted Bianchi VI_0 self-similar model [5, 6].

The Bianchi II solution found by Hewitt is *unique* amongst the tilted Bianchi models which prossess a G_2 (abelian) subgroup acting orthogonally transitively [4]. Nevertheless, because Bianchi type VIII and IX do not have this property, the question if tilted perfect fluid Bianchi type VIII and IX models admit a proper HVF is still open.

In this paper we prove that the answer to this question is negative, that is, tilted perfect fluid Bianchi type VIII and IX models do not admit a proper HVF.

It is important to state clearly our assumptions, which are as follows:

a) The spacetime manifold admits a G_3 group of isometries acting simply transitively on 3-dimensional spacelike surfaces $\mathcal{O}$. It is well known [7] that the unit normal u^a ($u^a u_a = -1$) to the surfaces of homogeneity $\mathcal{O}$ is geodesic and rotation free i.e. $u_{[a;b]} = \omega_{ab} = 0$.

b) The matter content of the SH model is tilted perfect fluid moving with 4-velocity $\tilde{u}^a$ ($\tilde{u}^a \tilde{u}_a = -1$) which is not orthogonal to the surfaces of homogeneity.

c) The SH model is *transitively self-similar*, that is, it admits a proper HVF **H** which, together with the Killing Vectors, generate a simply transitive homothety group of transformations of the spacetime manifold.

2   Self-similar Bianchi models

To establish the relation between the dynamic quantities defined by the observers $u^a, \tilde{u}^a$ we consider the 1+3 decomposition of the energy momentum tensor induced by each of them [8]. For the field u^a one has:

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a} u_{b)} + \pi_{ab} \tag{2}$$

where the dynamical quantities μ, p, q_a and π_{ab} are defined as follows:

$$\mu = T_{ab} u^a u^b, \;\; p = \frac{1}{3} T_{ab} h^{ab}, \;\; q_a = -h_a^c T_{cd} u^d, \pi_{ab} = h_a^c h_b^d T_{cd} - \frac{1}{3}(h^{cd} T_{cd}) h_{ab} \tag{3}$$

and $h_{ab} = g_{ab} + u_a u_b$ being the projection tensor associated with u^a.

For the tilted 4-velocity $\tilde{u}_a$ we have assumed that:

$$T_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} \tilde{h}_{ab} \tag{4}$$

where $\tilde{\mu}, \tilde{p}$ are the energy density and isotropic pressure respectively, measured by the tilted observers $\tilde{u}^a$ and $\tilde{h}_{ab} = g_{ab} + \tilde{u}_a \tilde{u}_b$ is the projection tensor of $\tilde{u}^a$.

Comparing (3) and (4) one obtains the following relations among the corresponding quantities [9]:

$$\mu = \tilde{\mu} + \Gamma^2 \upsilon^2 (\tilde{\mu} + \tilde{p})$$

$$p = \tilde{p} + \frac{1}{3} \Gamma^2 \upsilon^2 (\tilde{\mu} + \tilde{p})$$

$$q_a = \Gamma^2 (\tilde{\mu} + \tilde{p}) \upsilon_a \tag{5}$$

$$\pi_{ab} = \Gamma^2(\tilde{\mu} + \tilde{p})\left(v_a v_b - \frac{1}{3}v^2 h_{ab}\right) \tag{6}$$

where:

$$\tilde{u}_a = \Gamma\left(u_a + v_a\right) \tag{7}$$

$$\Gamma = \left(1 - v^2\right)^{-\frac{1}{2}}$$

and $v^2 = v^a v_a$, $u^a v_a = 0$.

We come now to the kinematical and the dynamical implications of the existence of the HVF. Concerning the kinematic part one has the following, easily established, result [10, 11, 12]:

Proposition 1

A spacetime admits a timelike HVF $H^a = Hn^a$ parallel to the unit timelike vector field n^a $(n^a n_a = -1)$ iff:

$$\sigma_{ab} = 0$$

$$\dot{n}_a = (\ln H)_{;a} + \frac{\theta}{3}n_a.$$

Moreover the homothetic factor ψ satisfies:

$$\psi = \frac{H\theta}{3}$$

where $\sigma_{ab} = \left(h_a^c h_b^d - \frac{1}{3}h_{ab}h^{cd}\right)n_{(c;d)}$ is the shear tensor, $\dot{n}_a = n_{a;b}n^b$ is the acceleration, $\theta = h^{cd}n_{c;d}$ is the rate of expansion and h_{ab} is the associated projection tensor of the timelike congruence n^a [8].

From Proposition 1 one has the useful result that if a SH model admits a proper HVF parallel to the fluid velocity $n^a = u^a$ then $\sigma_{ab} = 0$. Because the timelike congruence u^a is also geodesic and irrotational, the (0α)-constraint equation and the H_{ab}-equation [8, 13] imply:

$$\frac{2}{3}h_b^a \theta^{;b} = q^a \text{ and } H_{ab} = 0$$

where H_{ab} is the magnetic part of the Weyl tensor.

Due to the spatial homogeneity of the Bianchi models $h_b^a \theta^{;b} = 0 = q^a$ therefore, from equation (7), it follows that $u^a = \tilde{u}^a$ and the fluid is necessarily non-tilted. Because the only such models are the FRW cosmological models [1], we conclude that *the only perfect fluid Bianchi spacetimes which admit a proper HVF parallel to u^a are the corresponding FRW models* [2]. It is interesting to note that the above result extends to the more general case of a proper CKV (i.e. $\psi_{;a} \neq 0$). Hence *perfect fluid Bianchi spacetimes do not admit proper CKVs or HVF parallel to the unit normal u^a, except their FRW analogues* which is in agreement with the result of Coley and Tupper [14].

Using the above arguments we prove the main result of the paper:

Proposition 2

There are no Bianchi type VIII and IX transitively self-similar tilted perfect fluid models.

Proof

Suppose that Bianchi type VIII, IX models are transitively self-similar i.e. they admit a homothety group H_4 with 4-dimensional orbits and generators $\{\mathbf{H}, \mathbf{X}_\alpha\}$ where $\{\mathbf{X}_\alpha\}$ (greek indices take the values $1, 2, 3$) are the generators of the G_3 group of isometries acting simply transitively on the spacelike hypersurfaces $\mathcal{O}$ and $\mathbf{H}$ is the HVF. The Jacobi identities applied to the vector fields $\{\mathbf{H}, \mathbf{X}_\alpha\}$ imply that $\mathbf{H}$ is invariant under G_3 [15, 16]:

$$[\mathbf{X}_\alpha, \mathbf{H}] = 0. \tag{8}$$

We decompose $\mathbf{H}$ along and perpendicular to $\mathbf{u}$ as follows:

$$H^a = Hu^a + Y^a \tag{9}$$

where $H = -H^a u_a$ and $u^a Y_a = 0$.

The commutator (8) gives $\mathbf{X}_\alpha(H)\mathbf{u} = [\mathbf{Y}, \mathbf{X}_\alpha]$. Because $\mathbf{Y}$ lies in the hypersurfaces of spatial homogeneity, it follows that $\mathbf{X}_\alpha(H) = 0$ and $[\mathbf{Y}, \mathbf{X}_\alpha] = 0$, that is, both $\mathbf{Y}$ and H are invariant under the action of G_3. Furthermore, since $\mathbf{H}$ is a HVF of the spacetime manifold and $\mathbf{u}$ is geodesic and irrotational, one has $[\mathbf{H}, \mathbf{u}] = -\mathbf{u}(H)\mathbf{u} + [\mathbf{Y}, \mathbf{u}] = -\psi\mathbf{u}$ where $\mathbf{u}(H) = \psi$ [11] from which it follows $[\mathbf{Y}, \mathbf{u}] = 0$ i.e. $\mathbf{Y}$ is invariant under $\mathbf{u}$.

Therefore from (9) we may write $\mathbf{H} = H\mathbf{u} + A^\alpha \eta_\alpha$ where η_α is the group-invariant basis [17] and A^α are *constants*. Because $[\mathbf{H}, \eta_\alpha] = \mathbf{0}$ it follows that $A^\alpha C_{\alpha\beta}^\gamma = 0$ where $C_{\alpha\beta}^\gamma$ are the structure constants of the Bianchi type VIII and IX group of isometries. Hence $A^\alpha = 0$ which, in turn, implies that $\mathbf{H} = H\mathbf{u}$ and the HVF $\mathbf{H}$ is parallel to $\mathbf{u}$. By virtue of the previous considerations we conclude that self-similar Bianchi VIII, IX models do not admit a tilted (or non-tilted) perfect fluid interpretation, unless the spacetime becomes a FRW spacetime in which case the Bianchi IX is the only permissible type.

A side result of Proposition 2 is that the *general solution* of the homothetic equations in Bianchi type VIII and IX spacetimes is:

$$ds^2 = -dt^2 + (\psi t)^2 g_{\alpha\beta} \omega^\alpha \omega^\beta \tag{10}$$

where $g_{\alpha\beta}$ are constants of integration and ω^α are the Bianchi VIII or IX invariant 1-forms [17].

In this case the HVF $H^a = \psi t \delta_t^a$ and the fluid interpretation of self-similar Bianchi VIII and IX spacetimes necessarily involve anisotropic stress i.e. $\pi_{ab} \neq 0$, whether the fluid is tilted or non-tilted.

We note that the structure of a four dimensional Lie Algebra [15] shows that amongst the Bianchi models sharing the property of the existence of a proper HVF, the "singular" behaviour of H_4 (equation (8)) appears only in Bianchi types VIII and IX. Therefore we expect that this result does not extend to the rest of Bianchi models and the method developed in this paper can be easily applied leading towards to the determination of the homothety group H_4 and the structure of tilted perfect fluid Bianchi spacetimes. This will be the subject of a subsequent work.

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